Advanced Mathematical Models & Applications
Vol.8, Special Issue, 2023, pp.365-375

COMPUTATIONAL METHOD FOR VALUING AMERICAN PUT OPTION UNDER STOCHASTIC VOLATILITY: NEURAL NETWORK APPROACH

El Kharrazi Zaineb[1*], **Saoud Sahar**[2], **Mahani Zouhir**[1]

[1]Engineering Sciences and Energy Management Laboratory, National School of Applied Science, Agadir, Morocco
[2]Technical Research Laboratory, Faculty of Applied Sciences, Agadir, Morocco

Abstract. The current study presents a new approach, for estimating the price of an American put option under stochastic volatility, that combines the explicit finite difference method with a radial basis function neural network. We consider a sequence of Radial Basis function Neural Network, where each network learns the difference of the price function according to the Gaussian basis function. Then, we improve the superiority of artificial neural networks by comparing their performance and findings in the literature with those produced by neural networks in one dimension, using the Black Scholes partial differential equation in a stochastic environment. As a consequence, numerical findings demonstrate that the Artificial Neural Network solver can considerably reduce both computation time and error training.

Keywords: American option, stochastic volatility, explicit finite difference, neural network, radial basis function.
AMS Subject Classification: 91G20, 91G30, 91G60, 39A50, 97R40.
Corresponding author: El Kharrazi, Zaineb, Engineering Sciences and Energy Management Laboratory, National School of Applied Science, Avenue Tamesoult Agadir 80000, Agadir, Morocco, Tel.: +212659699392,
e-mail: *zaineb.elkharrazi@edu.uiz.ac.ma*
Received: 09 November 2021; Revised: 05 February 2022; Accepted: 08 March 2022;
Published: 30 November 2023.

1 Introduction

In the finance world, derivatives have become more and more popular and essential Fu et al. (2001); Speranda et al. (2015). Options, for example, are one of the most basic and important derivatives that actively traded on many exchanges, for that, the option valuation receives high attention. The holder of the option can sell or buy the underlying asset, such as a quantity of a company's stock, at a predetermined expiration date and a predefined price fixed in contract Longstaff & Schwartz (2001); Natenberg & Cohen (1994); Racicot & Théoret (2006). A put (call), option gives the owner the right to sell (buy) the underlying asset, and the price option differs according to the types of options: European options, American options, Asian options, Barrier options, Bermudan options...etc. Unlike the European option, which can be exercised just at maturity, American options can be exercised at any time until maturity, thence the valuation of this type of option does not have an exact form solution.

Therefore, for the valuation of financial derivatives, Black Scholes model developed by Fisher Black, Robert Merton, and Myron Scholes in 1973 Black & Scholes (2003), proposed the famous Black Scholes formula for pricing options, which considered as one of the most effective methods for determining accurate prices of options and still widely used in financial markets today. Thus, 1973 considered as the starting point for the intensive development of research concerning valuation derived products. During the 80s and 90s, the hedging of these products made market actors aware of the risk Dionne (2013); Glasserman (2004).

It is well-known that the volatility under Black Sholes model assumed to be constant, however from empirical studies based on real market data, one can observe that volatility is not constant in the real finance world. To treat this unrealistic phenomenon many approaches have been proposed.

After the famous work of Black-Scholes and Merton that assumes constant interest rate and constant volatility, several stochastic volatility models were introduced, such as Heston model Heston et al. (1993) and Hull–White model Hull et al. (1987), constant elasticity of variance model by Cox Cox et al. (1996), a multiscale stochastic volatility model by Fouque et al. Fouque et al. (1998), and a Lévy model by Carr et al. Carr et al. (2003). Compared with the simple model where the volatility is constant, the stochastic volatility model has a set of random variables, which augments the challenge to give the analytic solution of option price.

In the previous studies, there is no closed solution to the partial differential equation (PDE) describing the American option price under stochastic volatility, for this reason, we aim to solve our problem numerically using finite difference method.

The finite difference method is a technical method for approximating the solution of differential equation provided by a system of derivatives. The fundamental idea, is to convert a partial differential equation into an algebraic equation that is discretized in space, or discretized in space and time.

The basic idea of the current work is to approximate the value of the American put option under stochastic volatility using explicit finite difference method combined with neural network, or more precisely radial basis function neural network (RBF-NN). In order to increase the adequacy and performance, we aim to adjust and reinforce the explicit difference method by a neural network. Moreover, Artificial neural networks have recently gained much attention due to their success in a number of different fields and applications including finance and especially pricing option.Kasabov et al. (2019); Malliaris & Salchenberger (1993). Artificial neural networks are information treatment models that developed as generalizations of mathematical models of human knowledge, inspiring from the working principles of the human brain Alanis et al. (2019); Al-Aradi et al. (2018); Cortese et al. (2019); Janson & Tysk (2006); Lyaqini et al. (2022); Gasimov et al. (2019); Nachaoui et al. (2021); Rasheed et al. (2021). The basic component of an artificial neural network is a neuron (ex: like biological neuron). A biological neuron may be modeled artificially to perform computation, and then the model is characterized as an artificial neuron. A neuron is the basic processing element in a neural network Yadav et al. (2015). Each neuron receives one or more input to produces output.

Latest innovations in data science, have demonstrated that even very nonlinear multi-dimensional functions may be effectively represented using deep learning approaches Bengio & LeCun (2007). Furthermore, there are several research articles published recently, that investigate the connections between numerical algorithms for PDEs and neural networks to provide a new mathematical neural building blocks for different purpose. Since the RBF-NN has been recognized as useful method in different fields such as approximate optimization, pattern recognition, signal processing and so on Ghazdali et al. (2015); Sun,C et al. (2015); Wan,C et al. (1999), we adapt in this paper, the RBF-NN into the pricing American option under stochastic volatility problem in one dimension, in order to find the optimal parameters that minimize the error.

The contribution of this modified method to mathematical and computational finance is to obtain a more sophisticated closed-real price under stochastic volatility environment. Considering an asset price process S with stochastic volatility ν, in order to implement finite difference technique we need a discretization grid for the stock price and variance, also for maturity. Then we use the explicit finite difference method to solve the partial differential equation and get the value of our option. After that, we use the solution of our partial differential equation as inputs for radial basis function neural network, which will be used to find the closed real prices with errors less and high speed of convergence than the previous method in literature.

The layout of this paper is as follows. In section 2, we described the stochastic volatility model and the explicit finite difference method. We described our approach to price the put option under stochastic volatility using RBF-NN combined with an explicit finite difference method in section 3. Finally, section 4 dedicated to numerical results; where we show the performance and the accuracy as well as the robustness of the proposed method. Conclusion is given in section 5.

2 Option Pricing using Black-Scholes model with Stochastic Volatility

In this section, we describe a model for pricing American options and we present a finite difference approach to evaluate our option. Black-Scholes equation and the associated Black-Scholes partial differential equation that has been discussed in the work of Black and Scholes (1973), is one of the most well-known results in quantitative finance Racicot & Théoret (2006); Brandimarte (2013). It is used to solve the price of various financial derivatives, including the American option. In the following, the price of an underlying asset is modelled in a more realistic way by assuming stochastic volatility.

2.1 Stochastic Volatility Model

The Heston model has become one of the most important and famous models, in financial mathematics approach to pricing options known as stochastic volatility modelling.

Let $V(s, v, T)$ be the price of an American put option, with time to maturity T written to a stock of price S and variance ν, with no dividend. The option has a strike K, the drift r (risk neutral) and volatility σ. The dynamic of the share price under risk-neutral measure is presented by the following stochastic differential equation

$$dS(t) = rSdt + \sqrt{\nu(t)}SdZ_1(t) \tag{1}$$

Where $Z_1(t)$ is a Wiener process.
Supposing that the volatility follows an Ornstein-Uhlenbeck process, and using Itô's lemma we get the formula proposed by Heston in 1993:

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu}dZ_2(t) \tag{2}$$

Where $\mathbb{E}[dZ_1, dZ_2] = \rho dt$, and $\rho \in [-1, 1]$

Heston model describes the movements of the asset price when the asset price and volatility follow a random Brownian motion. We define $\theta > 0$ as the long term variance, $\kappa > 0$ as the threat of mean reversion.

It is more convenient to convert correlated Wiener process to an independent Wiener process. To do so, we apply Cholesky decomposition to Wiener process to get independent process W_1, W_2 under risk-neutral measure:

$$\begin{aligned} dZ_1 &= dW_1 \\ dZ_2 &= \rho dW_1 + \sqrt{1-\rho}dW_2 \end{aligned}$$

Then the dynamic of S and ν can be written as follow:

$$\begin{aligned} dS(t) &= rSdt + \sqrt{\nu(t)}SdZ_1(t) \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\rho\sqrt{\nu}dW_1 + \sigma\sqrt{\nu}\sqrt{1-\rho}dW_2 \end{aligned}$$

According to the application of the non-arbitrage pricing argument to Black-Scholes-Merton (1973) partial differential equation, then by applying the Itô's formula, a two-dimensional parabolic partial differential inequality can be derived for the value of our asset $V(S_t, \nu_t, t)$ using the previous stochastic volatility model:

$$\begin{aligned} \mathcal{L}V &= \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma\nu S \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\sigma^2 \nu \frac{\partial^2 V}{\partial \nu^2} + rS\frac{\partial V}{\partial S} \\ &+ [\kappa(\theta - \nu(t) - \lambda(S,\nu,t))]\frac{\partial V}{\partial \nu} - rV + \frac{\partial V}{\partial t} = 0 \end{aligned} \tag{3}$$

Where $\lambda(S,\nu,t)$ represent the price of volatility risk, which is independent of particular asset, and it is commonly chosen as $\lambda(S,\nu,t) = \lambda_0\ nu$ for some constant λ_0. Throughout the paper, and with no loss of generation, we assume that$\lambda_0 = 0$ as has been done in many previous studies like in Oosterlee et al. (2003).

In addition, an American option provides their holders the right to exercise at any time until maturity. For this reason, the early exercise possibility should be taken into account, in order that the price V has to be at least the same as the payoff function g :

$$V(S,\nu,t) \geqslant g(S,\nu) \tag{4}$$

For a put option the payoff function is $g(S,\nu) = max(K - S, 0)$.
Otherwise, the American option price can be defined as the solution of the linear complementarity problem:

$$\begin{cases} \mathcal{L}V.(V-g) = 0 \\ \mathcal{L}V \geqslant 0, \quad V \geqslant g \end{cases} \tag{5}$$

in a domain $\Omega = \{(S,\nu,t) | S \geqslant 0, \nu \geqslant 0, t \in [0,T]\}$.The initial and boundary conditions become the price at maturity is given by the payoff g. Also, it defines the initial conditions:

$$\begin{aligned} V(S,\nu,T) &= g(S,\nu) \\ V(0,\nu,t) &= g(0,\nu) \\ \lim_{s\longrightarrow\infty} \frac{\partial V}{\partial S} &= 0 \\ \lim_{\nu\longrightarrow\infty} \frac{\partial V}{\partial \nu} &= 0 \end{aligned} \tag{6}$$

2.2 Finite Difference Scheme

In the following subsection, we introduce the explicit finite difference method to obtain the American put option by solving the equation (3). For that, we need to discretized the region of definition Ω of our PDE, since if we use more grid points it will be more points where the function value is approximated. We consider a two-dimensional grid representing the stock price and the volatility.

In the following, we present the central difference approximation to the derivatives under a non-uniform grid.

Let $V_{i,j}^n = V(S_i, \nu_j, t_n)$ the American put value at t_n when the stock price is S_i and the volatility is ν_j, yet we write simply $V(S_i,\nu_j)$ for $V_{i,j}^n$.
So all partial differentiation could be stated as following:

First-order derivatives:

$$\frac{\partial V}{\partial S}(S_i,\nu_j) = \frac{V_{i+1,j}^n - V_{i-1,j}^n}{2ds} \tag{7}$$

$$\frac{\partial V}{\partial \nu}(S_i,\nu_j) = \frac{V_{i,j+1}^n - V_{i,j-1}^n}{2d\nu} \tag{8}$$

Second-order derivatives:

$$\frac{\partial^2 V}{\partial S^2} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{ds^2} \tag{9}$$

$$\frac{\partial^2 V}{\partial \nu^2} = \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{d\nu^2} \tag{10}$$

$$\frac{\partial^2 V}{\partial S \partial \nu} = \frac{V_{i+1,j+1} + V_{i-1,j-1} - V_{i-1,j+1} - V_{i+1,j-1}}{4dsd\nu} \tag{11}$$

With $i = 0, 1, ..., I$ ($I + 1$ points for the stock price)
$j = 0, 1, ..., J$ ($J + 1$ points for volatility)
$n = 0, 1, ..., M$ ($M + 1$ points for the maturity)

In order to find the value of our option we use the θ-method. This method combines the forward and backward scheme with the help of a scaling parameter and it is defined via the relationship:

$$\frac{V^{n+1} - V^n}{dt} = L(\theta V^{n+1} + (1 - \theta)V^n) \tag{12}$$

Since we based on an explicit finite difference framework, we set $\theta = 0$, then the equation (12) become:

$$V^{n+1} = V^n + dtLV^n \tag{13}$$

with L is a matrix based on the operator defined in equation (3) After substituting the approximations to the derivatives, we obtain:

$$\begin{aligned} V^{n+1}_{i,j} &= a^n_{i,j}V^n_{i,j} + b^n_{i,j}V^n_{i-1,j} + cV^n_{i+1,j} + dV^n_{i,j-1} + eV^n_{i,j+1} \\ &+ f(V^n_{i+1,j+1} + V^n_{i-1,j-1} - V^n_{i-1,j+1} - V^n_{i+1,j-1}) \end{aligned} \tag{14}$$

where:

$$\begin{aligned} a^n_{i,j} &= 1 - dt(i^2\nu_j + \frac{\sigma^2 j}{d\nu} + r) \\ b^n_{i,j} &= \frac{idt}{2}(i\nu_j - r) \\ c^n_{i,j} &= \frac{idt}{2}(i\nu_j + r) \\ d^n_{i,j} &= \frac{dt}{2d\nu}(\sigma^2 j - \kappa(\theta - \nu_j)) \\ e^n_{i,j} &= \frac{dt}{2d\nu}(\sigma^2 j + \kappa(\theta - \nu_j)) \\ f^n_{i,j} &= \frac{ijdt\sigma}{4} \end{aligned} \tag{15}$$

After approximating the solution by using the numerical operators of the function's derivatives and finding the solution at specific grids, we will introduce our modified method which based on radial basic function neural network algorithm presented in Sahar & Zouhir (2022).

3 The Modified Method: RBFNN-FDM for pricing option

3.1 Finite Difference Radial Basis Function Neural Network for pricing American option

It is well known that the finite difference method is one of the most popular methods of numerical solution of partial differential equations. Our modified method is a hybrid method based on artificial intelligence and explicit finite difference method. As we treat in the previous section, the volatility is considered stochastic, hence there is not an exact solution for our pricing problem. Although we can always use some numerical methods in order to approximate and find a closed solution for such problems.

We chose the explicit finite difference method because it can be modified to allow for the pricing of American puts, which requires a reformulation of the boundary conditions. After getting the approximated prices, we enter them into the radial basis function neural network as a second step which will be used to find the closed real prices.

The basic idea here is to relate Black-Scholes under stochastic volatility model with machine learning from an optimization point of view. To estimate the model parameters we have to find the value of neural network's hidden units so that output matches the observed option prices with errors less than the classical method.

Our choice is based on the fact that in the RBF network, it is essential to set correct initial states; while other methods (for example MLP networks) use randomly generated parameters initially, and since our input are not random because we obtained them from explicit finite difference result, it seems that RBF neural network is more suitable in our case.

3.2 Pricing Algorithm

We choose in this paper to combine the Finite Difference method with neural network RBF, in order to find the price of our put option in the shortest time and with a small error. The key of these combined methods is that the output of finite difference becomes the input of RBF networks, to get the optimal weight and bias. On the other hand, the activation function of the RBF-NN computes the Euclidean norm (based on the Gaussian transformation) between the signal from the input vector and the center of that unit.

Therefore, the proposed method consists of two main steps. First is the training step, in which we solve the explicit finite difference equation (15), so the data that we will get from this method will be the inputs of RBF-NN to find the optimal weight and bias. The second step is testing; which requires only the optimal weight and bias to find a closed and real result, also to optimize our profit also to minimize our losses.

The following figure represents the general process of the proposed method:

Otherwise, we introduce the algorithm we based on to create the architecture of our proposed method, RBF neural network, in the training and testing stage:

Algorithm: Radial Basic Function Neural Network

- Initialize the parameters of Black Scholes equation under stochastic volatility $r, T, K,\ S_0, S_{max},\ S_{min}, \sigma, I, J, M, \kappa, \theta, \rho, \nu_0$

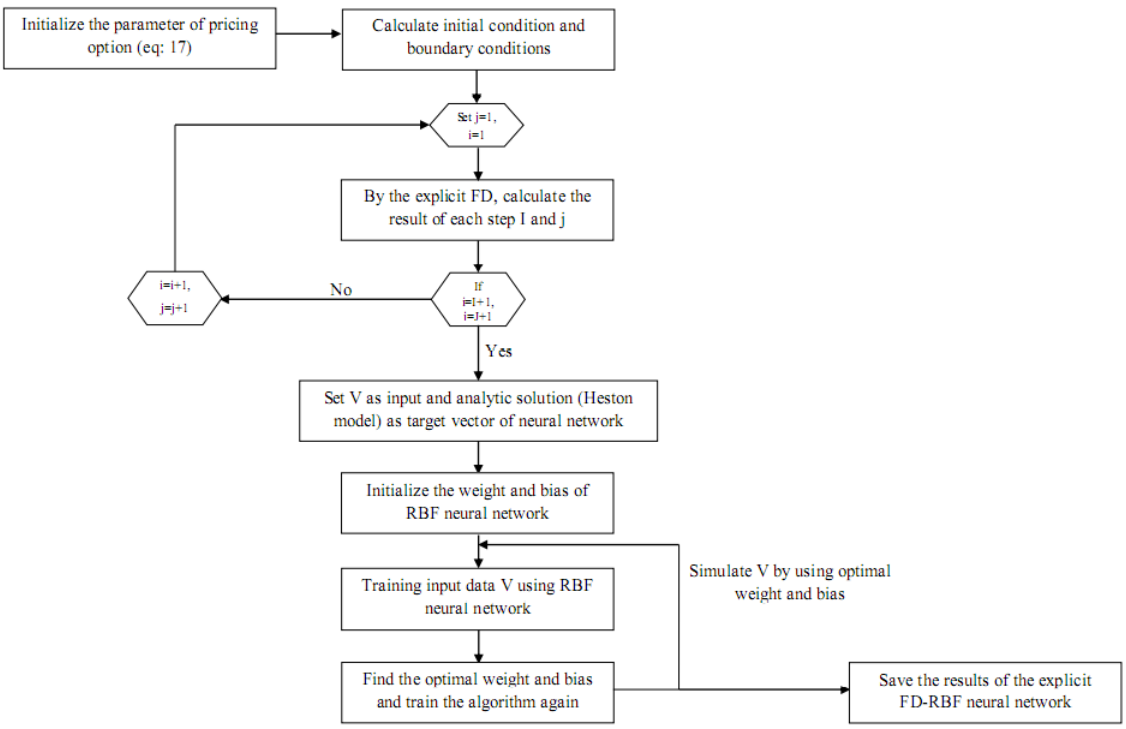
- Calculate initial conditions $log(\dfrac{S_{min}}{K}), log(\dfrac{S_{max}}{K})$ and semi analytic solution, taking on consideration the boundary condition (6)

$$\begin{cases} V(t,s) \geqslant G(s) & (t,s) \in [0,T] \times \mathbb{R} \\ V(T,s) = G(s) & s \in \mathbb{R} \end{cases}$$

- By explicit finite difference method calculate the results $V_{i,j}$ of each level in
- Set $V_{i,j}$ as input and semi analytic solution as target vector of neural network
- Initialize RBF-NN parameters (weight and bias) randomly
- Training input data by using RBF-NN to find the optimal weight and bias
- Simulate the input $V_{i,j}$ data with target by using optimal weight and bias

For testing the reliability of our algorithm, we will present in the next section some numerical

Figure 1: General Representation of the Proposed Modified Method



results of an American put option under stochastic volatility.

4 Numerical Experiments and Results

In this section, we present numerical results for American put options under stochastic volatility to show the accuracy and efficiency of our modified method. All simulations are performed on a Windows 10 machine with a processor Intel Core i5, 2.5 GHz and 8 GB RAM using MATLAB R2019.

In the first stage, we compare the value of the American put option under stochastic volatility model obtained by finite difference-RBFNN, with some previous existing methods. Then we will study the error and CPU time. CPU time (central processing unit), is one of the most important parameters that prove the performance of such a method (or a code). For this reason, we calculate the CPU time at each step of calculation our price put option, and the error training as well.

4.1 American put Option valuation

To compare our method with existing methods in the literature, we have the parameters listed in table (1)

Table 1: Parameter set for the Example 5.1

κ	θ	σ	ρ	K	$\sqrt{\nu_0}$	T	r
5	0.16	0.9	0.1	10	0.25	0.25	0.10

and we initialize the step sizes: $I = 125$, $J = 6$, and $M = 25$ In this study we supposed that the market condition has been verified, so the table 2 described the different values of

Table 2: American put option under stochastic volatility

S_0	8	9	10	11	12
FD-RBFNN (our method)	1.9984	1.1090	0.5158	0.2096	0.0847
VN	1.9970	1.1112	0.5254	0.2201	0.0875
ZFV	2.0000	1.1076	0.5202	0.2138	0.0821
IT-PSOR	2.0000	1.1074	0.5190	0.2130	0.0818
OO	2.0000	1.1070	0.5170	0.2120	0.0815
CP	2.0000	1.1080	0.5316	0.2261	0.0907

the put option while changing the value of the stock price (i.e. underlying asset). As shown in the table above, we compare our method with five results that deal with American put option under stochastic volatility, obtaining by Vellekoop and Nieuwenhuis (2009)(denoted by VN) ,Zven ,Forsyth, Vetzal(1998)(denoted by ZFV), Ikonen and Toivanen (2007) or IT-PSOR (Ikonen and Toivanen PSOR method), Oosterlee (2003) denoted by (OO) and Clarke and Parrott (1999)(denoted by CP).

It is apparent that the solutions obtained by FD-RBFNN are closed to the solutions of the majority of the presented methods. If we take for example VN method, we observe from table (3) that the error produces is small. So the FD-RBFNN method gives us a closed solution of the pricing American option under stochastic volatility.

Table 3: Error between FD-RBFNN and VN

S_0	8	9	10	11	12
Error	-0.0014	0.0022	0.0096	0.0105	0.0028

In order to prove the efficiency of the proposed method, we calculate the CPU time. We can see that FD-RBFNN seems to be more faster than the previous study (cpu=0.16 see). So, RBF neural networks have the priority to work with since one of our goals is to speed up the calculation time to enhance the American option.

Table 4: CPU time(sec)

S_0	8	9	10	11	12
cpu	0.0397	0.0656	0.0403	0.0435	0.0515

In order to reinforce the explicit finite difference RBF-NN we propose to compare our method with some famous stochastic volatility models such as Stein-Stein, Hull-White and 3/2 Model. For this purpose, we consider the following parameters:

Table 5: General parameters

T	r	S_0
0.50	0.05	100

We have also some specific parameters for the chosen methods from literature

Table 6: Parameter in specific cases

Model	σ	κ	ρ	θ	ν_0	a_ν
Stein-Stein	0.18	2	-0.5	0.18	0.22	-
Hull-White	0.10	-	-0.7	-	0.03	0.03
3/2 Model	0.12	2	-0.3	0.04	0.035	-

The result obtained in table (7) improves the efficiency and accuracy of the RBF-NN method since the error calculated with the other methods is small, and the prices are more closed to the analytical Heston model. Hence, what makes the RBF-NN better than classic methods is the speed of convergence (CPU time = 0.0417 sec). Therefore, the proposed approach provides a better perspective to describe the behavior of the option pricing model represented by the Black-Scholes equation with stochastic volatility and can be preferred due to reliability and accuracy with minimal computational effort.

Table 7: American put option under stochastic volatility

K	90	95	100	105
FD-RBFNN (our method)	1.4996	2.0623	3.9412	6.7921
Stein-Stein	1.8739	3.1445	5.0055	7.5848
Hull-White	0.9070	2.0388	3.9329	6.7195
3/2	1.1185	2.3542	4.3233	7.1132

5 Conclusion

In this work, we have proposed a neural network approach to price the American put option under stochastic volatility. We have combined the neural network to solve the Black Scholes partial differential equation with explicit finite difference method using radial basis function. Our objective in the first place was to compare the performance and the results exist in previous studies with radial basis function neural network. Although the results were closed, the neural network value was closer to the semi-analytic solution. As a result, the suggested approach provides a better perspective to describe the behavior of the option pricing model, and may be chosen owing to its reliability and accuracy while requiring a minimal computational effort.

We believe that the numerical techniques presented in this paper makes it promising to be extended in two-dimensions and also to be used in other similar fractional models for pricing different options.

References

Alanis, A.Y., Arana-Daniel, N., & Lopez-Franco, C. (Eds.). (2019). *Artificial Neural Networks for Engineering Applications.* Academic Press.

Al-Aradi, A., Correia, A., Naiff, D., Jardim, G., & Saporito, Y. (2018). Solving nonlinear and high-dimensional partial differential equations via deep learning. arXiv preprint arXiv:1811.08782.

Black, F., Scholes, M. (1973). The valuation of options and corporate liabilities. *Journal of Political Economy, 81*(3), 637-654.

Bengio, Y., LeCun, Y. (2007). Scaling learning algorithms towards AI. *Large-scale kernel machines, 34*(5), 1-41.

Brandimarte, P. (2013). *Numerical methods in finance and economics: a MATLAB-based introduction.* John Wiley & Sons.

Carr, P., Geman, H., Madan, D.B., & Yor, M. (2003). Stochastic volatility for Lévy processes. *Mathematical finance, 13*(3), 345-382.

Cortese, A., De Martino, B., & Kawato, M. (2019). The neural and cognitive architecture for learning from a small sample. *Current Opinion in Neurobiology,* 55, 133-141.

Cox, J.C. (1996). The constant elasticity of variance option pricing model. *Journal of Portfolio Management*, 15.

Dionne, G. (2013). Risk management: History, definition, and critique. *Risk Management and Insurance Review, 16*(2), 147-166.

Fouque, J.P., Papanicolaou, G., & Sircar, K.R. (1998). Asymptotics of a two-scale stochastic volatility model. Equations aux derivees partielles et applications, in honour of Jacques-Louis Lions, 517-525.

Fu, M.C., Laprise, S.B., Madan, D.B., Su, Y., & Wu, R. (2001). Pricing American options: A comparison of Monte Carlo simulation approaches. *Journal of Computational Finance, 4*(3), 39-88.

Gasimov, Y., Pashayev, A., Azizov, B., Agamalieva, L., Lacinová, V., & Šmerek, M. (2019). Application of decision-making expert systems in transportation. *NAŠE MORE: znanstveni časopis za more i pomorstvo, 66*(3), 130-134.

Ghazdali, A., Hakim, A., Laghrib, A., Mamouni, N., & Raghay, S. (2015). A new method for the extraction of fetal ECG from the dependent abdominal signals using blind source separation and adaptive noise cancellation techniques. *Theoretical Biology and Medical Modelling, 12*(1), 1-20.

Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering* (Vol. 53, pp. xiv+-596). New York: Springer.

Janson, S., Tysk, J. (2006). Feynman–Kac formulas for Black–Scholes-type operators. *Bulletin of the London Mathematical Society, 38*(2), 269-282.

Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies, 6*(2), 327-343.

Hull, J., White, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal of Finance, 42*(2), 281-300.

Kasabov, N.K. (2019). *Time-space, spiking neural networks and brain-inspired artificial intelligence.* Heidelberg: Springer.

Longstaff, F.A., Schwartz, E.S. (2001). Valuing American options by simulation: a simple least-squares approach. *The Review of Financial Studies, 14*(1), 113-147.

Lyaqini, S., Nachaoui, M., & Hadri, A. (2022). An efficient primal-dual method for solving non-smooth machine learning problem. *Chaos, Solitons & Fractals,* 155, 111754.

Malliaris, M., Salchenberger, L. (1993). A neural network model for estimating option prices. *Applied Intelligence, 3*(3), 193-206.

Nachaoui, A., Nachaoui, M. & Gasimov. B. (2021). Parallel numerical computation of an analytical method for solving an inverse problem. *Advanced Mathematical Models & Applications, Jomard Publishing, 6*(2), 162-173.

Rasheed, S. M., Nachaoui, A., Hama, M. F. & Jabbar, A. K. (2021). Regularized and preconditioned conjugate gradient like-methods methods for polynomial approximation of an inverse Cauchy problem. *Advanced Mathematical Models & Applications, 6*(2), 89-105.

Natenberg, S., Cohen, J.M. (1994). Option volatility & pricing: advanced trading strategies and techniques.

Oosterlee, C.W. (2003). On multigrid for linear complementarity problems with application to American-style options. *Electronic Transactions on Numerical Analysis, 15*(1), 165-185.

Racicot, F.É., Théoret, R. (2006). Finance computationnelle et gestion des risques. PUQ.

Šperanda, I., Tršinski, Z. (2015). Hedging as a business risk protection instrument. *Ekonomski vjesnik: Review of Contemporary Entrepreneurship, Business, and Economic Issues, 28*(2), 551-565.

Sun, C., Jin, Y., Zeng, J., & Yu, Y. (2015). A two-layer surrogate-assisted particle swarm optimization algorithm. *Soft computing, 19*(6), 1461-1475.

Wan, C., Harrington, P.D.B. (1999). Self-configuring radial basis function neural networks for chemical pattern recognition. Journal of Chemical Information and Computer Sciences, *39*(6), 1049-1056.

Yadav, N., Yadav, A., & Kumar, M. (2015). *An introduction to neural network methods for differential equations* (pp. 13-15). Netherlands: Springer.

Sahar, S., Zouhir, M. (2022). Pricing American Put Option using RBF-NN: New Simulation of Black-Scholes. *Moroccan Journal of Pure and Applied Analysis, 8*(1), 78-91.